Microscopic structure of a vortex line in superfluid neutron star matter

F.V. De Blasio and Ø. Elgarøy
Department of Physics, University of Oslo, N-0316 Oslo, Norway
(February 9, 2008)

The microscopic structure of an isolated vortex line in superfluid neutron matter is studied by solving the Bogoliubov-de Gennes equations. Our calculation, which is the starting point for a microscopic calculation of pinning forces in neutron stars, shows that the size of the vortex core varies differently with density, and is in general smaller than assumed in some earlier calculations of vortex pinning in neutron star crusts. The implications of this result are discussed.

PACS numbers: 26.60.+c, 21.65.+f, 97.60.Jd, 97.60.Gb, 74.26.Bt

The interior of a neutron star constitutes the only known physical system close to infinite nuclear and neutron matter. Most efforts to describe this system have mainly focussed on the role of nucleonic interactions at zero and finite temperature, while less attention has been paid to a microscopic description of excited states of the system, including superfluid vortex lines induced by the rotational state of the star. On the other hand, there are important observables of astrophysical relevance that might be influenced by the presence of vortex lines. A mutual check of the nuclear many body physics and of the theory of neutron star interiors comes for example from the study of pulsar glitches, sudden increases in the spinning frequency of the crust of the pulsar, followed by a slower tendency to conditions close to the original ones. It is thought that glitching events represent a direct manifestation of the presence of superfluid vortices in the interior of the star [1], the triggering event being an unbalance between the hydrodynamical forces acting on the vortex and the force of interaction of the vortex with the nuclei present in the crust (pinning force). There have been quite large uncertainties regarding the value of the pinning force, leaving room for quite opposite views about the validity of the vortex pinning model [2]. One source of uncertainty is related to the value of the pairing energy gap in uniform neutron matter, a quantity strongly dependent on the value of the neutron particle-particle matrix element near the Fermi surface [3–5]. A second problem is due to the very approximate way of treating vortex states in neutron matter. Usually a vortex is seen as a cylinder of normal matter (the vortex core) of radius equal to the BCS coherence length $\xi_0 \approx 0.8\hbar^2 k_F/2m\Delta$ where k_F is the Fermi wavenumber, m is the neutron mass and Δ is the neutron pairing gap [6–8]. The pinning of the vortex to the nucleus, also treated as a classical object, is due to the loss of superfluidity that occurs when the two objects superimpose.

Several experiments and theoretical calculations in type II superconductors show that a vortex is a quite complicated quantum object [9]. Bound states that can be formed in the center of the core as revealed by the increase in the density of states (investigated by scanning-tunneling microscopy) may change the local pairing gap in the core and the vortex size. In addition, they show that superfluidity is not completely suppressed in the core. The latter result can be in principle predicted also with the Ginzburg-Landau theory (GL), but unfortunately the GL theory is not reliable far from the transition temperature, which is the case occurring in neutron stars.

In the present Letter we propose to study the structure of a vortex in superfluid neutron matter using a microscopic, fully quantum-mechanical approach. More specifically, we shall make use of the Bogoliubov-de Gennes equations [10], that have been successfully employed to study vortices in type II superconductors [11,12] and more in general non-homogeneous superconductivity. The use of a model well-tested in the laboratory represents, in our view, a major merit of this kind of approach. There are however important differences between vortices in neutron matter and those in superconductors as studied in Ref. [11,12] that make a direct application of the previous findings impossible, and require a new *ab-initio* calculation. The first obvious difference is in the basic constants setting the scale of the problem, like the mass of the particles and the density regime. The difference is not merely quantitative, since in our case the parameter $\xi_0 k_F$ defining the importance of quantum effects is smaller than in the type-II superconductors, resulting in the possibility of strong deviations from previous solid-state calculations and also from semiclassical methods. Secondly, due to the charge neutrality of neutrons, magnetic fields do not play any role in our calculation and vortices are generated by the rotational state of the star. A further difference is that the temperature in superconductors is an important variable of the problem, while in our case, due to the high value of the Fermi temperature compared to the interior temperature of a neutron star, it can be set equal to zero. A fourth major difference is the role of the inter-particle interaction, which in the case of neutron matter is strongly density-dependent. Finally, we shall be mostly interested in information about the pinning force, which depends in a delicate manner on the length scale of the vortex core.

A vortex state in neutron matter can be described by the Bogoliubov-de Gennes (BdG) equations [10]:

$$\left(\frac{\mathbf{p}^2}{2m}+W(\mathbf{r})-E_F\right)U_i(\mathbf{r})+\Delta(\mathbf{r})V_i(\mathbf{r})=E_iU_i(\mathbf{r}) \quad (1)$$

$$-\left(\frac{\mathbf{p}^2}{2m}+W(\mathbf{r})-E_F\right)U_i(\mathbf{r})+\Delta^*(\mathbf{r})V_i(\mathbf{r})=E_iV_i(\mathbf{r}) \quad (2)$$

where $\mathbf{p}=-i\hbar\nabla$, U and V are the quasiparticle amplitudes, E_F is the Fermi energy, $\Delta(\mathbf{r})$ is the (space-dependent) pairing gap, m is the neutron mass, $W(\mathbf{r})$ is an external potential due to e.g. a nucleus, and the subscript i represents all relevant quantum numbers. We write the quasiparticle states as

$$U_i(\mathbf{r})=\frac{1}{L^{1/2}}u_{n\mu}(\rho)\exp[i(\mu-1/2)\theta]\exp[i(k_zz)] \quad (3)$$

$$V_i(\mathbf{r})=\frac{1}{L^{1/2}}u_{n\mu}(\rho)\exp[i(\mu+1/2)\theta]\exp[i(k_zz)]. \quad (4)$$

Here L is the length of the cylinder, taken equal to 1 in our calculations since all quantities are per unit length of the vortex, n is a radial quantum number and μ is half an odd integer. The quantity k_z is the (conserved) momentum in the z direction. The pairing gap has to be calculated self-consistently as

$$\Delta(\mathbf{r})=-g\sum_{i;0<E_i<\hbar\Omega}U_i(\mathbf{r})V_i^*(\mathbf{r}) \quad (5)$$

where the sum is over quasiparticle states having energy E_i smaller than a cutoff $\hbar\Omega$, and $g>0$ is the pairing strength. The meaning of these parameters is quite different for neutron matter compared with the solid state case, and will be discussed later on. Due to the above angular dependence of the quasiparticle states, the pairing gap has the form $\Delta(\mathbf{r})=\Delta(\rho)\exp[-i\theta]$, corresponding to a vortex with one quantum of circulation. Following [11] we expand the quasiparticle states in terms of cylindrically symmetric Bessel functions and impose the boundary condition of zero gap at the edges of a cylinder of radius R that is chosen sufficiently large. More specifically, the basis functions are chosen as

$$\phi_{jm}(\rho)=\frac{\sqrt{2}}{RJ_{m+1}(\alpha_{jm})}J_m\left(\alpha_{jm}\frac{\rho}{R}\right),\quad j=1,\ldots,N, \quad (6)$$

where $m=\mu\pm\frac{1}{2}$ is an integer and α_{jm} is the jth zero of the Bessel function $J_m(x)$. The dimension N of the basis is chosen large enough to ensure convergence and stability of the quantities of interest. The quasiparticle amplitudes are expanded as

$$u_n(\rho)=\sum_j c_{nj}\phi_{j\mu-\frac{1}{2}}(\rho) \quad (7)$$

$$v_n(\rho)=\sum_j d_{nj}\phi_{j\mu+\frac{1}{2}}(\rho). \quad (8)$$

For a given value of μ Eqs. (1,2) can then be written as a $2N\times 2N$ matrix eigenvalue problem

$$\begin{pmatrix} T^- & \Delta \\ \Delta^T & T^+ \end{pmatrix}\Psi_n=E_n\Psi_n, \quad (9)$$

where the superscript T denotes the transpose of a matrix, $\Psi_n^T=(c_{n1},\ldots,c_{nN},d_{n1},\ldots,d_{nN})$,

$$T^\pm=\frac{\hbar^2}{2m^*}\left(\frac{\alpha^2_{j\mu\pm1/2}}{R^2}+k_z^2-k_F^2\right)\delta_{jj'},$$

and the matrix Δ is given by

$$\Delta_{jj'}=\int_0^R\phi_{j\mu-1/2}(\rho)\Delta(\rho)\phi_{j'\mu+1/2}(\rho)\rho d\rho.$$

We have in these equations set the mean field $W(\mathbf{r})$ equal to zero, corresponding to an isolated vortex line in infinite neutron matter. Interactions between neutrons set up an internal mean field which can be simulated by replacing the neutron mass m by an effective mass m^*. However, for dilute neutron matter $m^*\approx m$ [4]. Starting from an appropriate approximation to $\Delta(\rho)$, we solve Eq. (9) for several values of μ to obtain the quasiparticle amplitudes U and V. A new approximation to $\Delta(\rho)$ is then obtained from Eq. (5). The procedure is iterated until convergence is achieved.

The BdG equations as written above have been specialized to the case of a point-like neutron-neutron interaction of the form $v(\mathbf{r},\mathbf{r}')=-g\delta(\mathbf{r}-\mathbf{r}')$. This is the same approximation as used in Refs. [10–12] and should be considered even more reliable in the case of neutron matter, where the range of the inter-particle interaction is of the order ~ 1 fm, and is thus larger than the length scale of the inhomogeneities, which is of the order of several fm. In dealing with a zero-range force, a cutoff energy $\hbar\Omega$ has to be introduced for the gap equation to converge. The density dependence of the pairing gap as calculated with finite-range forces, see for example [4], can be easily translated to a dependence of g on the density. We shall thus fix $\hbar\Omega=50$ MeV and use the value of g that reproduces the gap at large distance from the vortex core as calculated for a homogeneous system.

Fig. 1 shows the pairing gap as a function of the distance from the vortex axis, $\Delta(\rho)$ using the above formalism and for three different values of the neutron Fermi wavenumber. In all cases the gap increases from a value zero to an asymptotic one Δ_∞. The latter quantity represents evidently the value of the gap as calculated for uniform neutron matter. In the figure we have fixed the value of the effective pairing strength g to reproduce the value as calculated from the bare Bonn A potential [4]. Like for vortices in type II superconductors, the gap is found to increase linearly for small values of ρ before reaching the asymptotic value. The linear rate of increase of the gap defines a coherence length that for consistency with ref. [11] we call ξ_2, given by

$$\lim_{\rho\to 0}\Delta(\rho)=\Delta_\infty\frac{\rho}{\xi_2}. \quad (10)$$

As visible from the figure this coherence length represents an appropriate length scale of the vortex core. A second length scale, the BCS coherence length ξ_0, can be defined as the distance from the vortex axis at which the energy gain of the Cooper pair due to the velocity field becomes of the order of the energy necessary to break the pair (or also by the condition that the superfluid velocity $v_s = \hbar/m\rho$ coincides with the Landau velocity for the suppression of superfluidity, $2\Delta_\infty m/\hbar k_F$), and it is found $\xi_0 = 16.88 k_F/\Delta_\infty$. This BCS coherence length, which microscopically represents the size of a Cooper pair [13], is the one usually employed to calculate the interaction of a vortex with the nuclei in the crust of a neutron star [6–8,14]. In particular, matter is supposed to be completely normal ($\Delta(\rho) = 0$) for $\rho < \xi_0$ and superfluid ($\Delta(\rho) = \Delta_\infty$) for $\rho > \xi_0$. In a pictorial view, this length can be obtained intersecting the gap at infinity, Δ_∞ with the curve $g(\rho) = 0.814 \times \hbar^2 k_F/2m\rho$. The resulting gap profile is shown in Fig. 2 for the cases $k_F = 0.1$ fm^{-1} and $k_F = 0.5$ fm^{-1}. It is evident that this definition can be a reliable estimate of the vortex core size only when the intersection occurs for values of ρ smaller than ξ_2. The plot for $k_F = 0.1$ fm^{-1} is a case where the intersection occurs where the gap has already saturated. According to the common view of neutron star pinning, all the matter at $\rho < \xi_0$ should be considered in a normal state. This is not the case, as clearly visible from the figure. For $k_F = 0.5$ fm^{-1}, Fig. 2 shows that $\xi_0 < \xi_2$, and the approximation above is probably not too bad, although it somewhat overestimates the gap in the region $\xi_0 < \rho < \xi_2$.

Table I shows ξ_0 and ξ_2 as functions of the Fermi momentum, where the pairing strength g has been fitted to the $^1S_0$ gap of the Bonn A potential [4]. It is visible from the table that the value of ξ_2 seems to be uncorrelated with ξ_0. In general, ξ_2 remains within values between 6 to 10 fm, while the values of ξ_0 are much more scattered. Due to the fact that Friedel oscillations may slightly distort the vortex structure at small distances from the core [12], we also studied some vortex length scales defined without making use of the behavior close to the axis, for example the distance from the vortex core at which the gap is a fraction of Δ_∞, typically 50 % and 90 %. As expected, these quantities are larger than ξ_2, but show a similar trend. Overall, we found these lengths to agree better with ξ_2 than with ξ_0.

To understand the scaling of the vortex size as a function of the Fermi momentum, we also performed calculations keeping the pairing strength fixed and changing only the Fermi wavenumber, even if this produced unrealistic gaps. We found ξ_2 to be a decreasing function of the wavenumber, a result which agrees qualitatively with previous work on vortices in superconductors [11,15] where the vortex size is predicted to scale as $\sim k_F^{-1}$, but is very different from the behavior of the BCS coherence length, which scales proportionally to k_F/Δ_∞. In the realistic case, where the pairing strength is fitted to the gap in uniform matter, the coherence length ξ_2 increases as a function of k_F. Thus, the density dependence of the pairing strength g is also very important, and the vortex size appears to depend in a quite complex way both on g and on the Fermi wavenumber. Studying the distribution of the eigenvalues, we have found indeed that several bound states are formed in the vortex core. This feature is different from the solid-state case [11] where essentially only one bound state at each value of the angular momentum μ is formed. The value of ξ_2 is probably affected in a complex way by all the bound states, making a simple scaling law difficult to work out. As a partial conclusion, however, we can state that when varying the Fermi wavenumber and the pairing strength over the range of values relevant for $^1S_0$ neutron pairing, the coherence length ξ_2 does not change by more than 50 %, while ξ_0 can vary by a factor of ten.

We are now able to speculate on the applications of our results to the case of vortex pinning in the crust. The use of ξ_2 instead of ξ_0 as the length scale of the vortex core, implies that the pinning regime is approximately the same in the whole inner crust. In particular the super-weak pinning, proposed in the case of very large vortex core and in which several nuclei may be enveloped on one single plane of the vortex, is at variance with our findings. A second consequence already predictable on the basis of the data presented is that the pinning force exerted by a nucleus on a vortex should scale like $\sim \Delta^2 k_F/\xi_2$. We expect an increase of the pinning force of the order $\sim \xi_0/\xi_2$ compared with the cases where ξ_0 is used for the size of the vortex core, and this can be a very large factor for low densities.

We also examined the distortion of a vortex line in the presence of a mean field potential of cylindrical symmetry. To simulate the presence of a nuclear cluster, we chose a Woods-Saxon shape for the potential $W(\mathbf{r})$ with nuclear parameters of the order of those expected in the inner crust of a neutron star. In presence of a potential field, the diagonal parts of the Hamiltonian matrix become $T^\pm \to T^\pm + W^\pm$ where the matrix elements of the potential matrix are of the form $W^\pm_{jj'} = \int d\rho\rho\phi_{j\mu\pm1/2}(\rho)W(\rho)\phi_{j'\mu\pm1/2}(\rho)\delta_{jj'}$. We found a decrease of the coherence length in presence of the field. This shows that the shape of the vortex line is sensitive to the presence of the nucleus, an effect that might have important consequences in vortex pinning. This in fact contrasts with the assumption implicit in many calculations of vortex pinning in neutron star crusts, where the vortex core is assumed to be inert during the pinning state.

A further advantage the BdG formalism, not exploited in the present study, is that the Hamiltonian is diagonalized, resulting in a consistent calculation of the energy, while in other approaches the local density approximation [17] or Ginzburg-Landau approaches [16] need to be used. The calculation of the pinning energy and force of a nucleus with a superfluid vortex, technically more involved but conceptually straightforward, is thus under consideration [18].

We thank N. Hayashi for providing information about the calculational details of Ref. [12]. F. V. De Blasio would also like to thank J. Paaske for fruitful discussions on vortices in superconductors, and NORDITA, Copenhagen for hospitality during the development of parts of this work. Ø. Elgarøy thanks A. Botnen for useful advice on the computational details.

---

[1] P. W. Anderson and N. Ithoh, Nature **256**, 25 (1975).
[2] P. B. Jones, Phys. Rev. Lett. **79**, 792 (1997).
[3] J. Wambach, T. L. Ainsworth and D. Pines, Nucl. Phys. **A555**, 128 (1993).
[4] Ø. Elgarøy, L. Engvik, M. Hjorth-Jensen and E. Osnes, Nucl. Phys. **A604**, 466 (1996).
[5] H.-J. Schulze, J. Cugnon, A. Lejeune, M. Baldo and U. Lombardo, Phys. Lett. **B375**, 1 (1996).
[6] M. A. Alpar, Astrophys. J. **213**, 527 (1977).
[7] T. Takatsuka, Prog. Theor. Phys. **71**, 1432 (1984).
[8] G. Lazzari and F. V. de Blasio, Z. Physik **A353**, 13 (1995).
[9] See e.g. various papers in Physica **B210** vol. 3/4 (1995).
[10] P. G. de Gennes, *Superconductivity of Metals and Alloys* (Addison-Wesley, Reading, MA, 1989).
[11] F. Gygi and M. Schlüter, Phys. Rev. **B43**, 7609 (1991).
[12] N. Hayashi, T. Isoshima, M. Ichioka and K. Machida, Phys. Rev. Lett. **80**, 2921 (1998).
[13] F. V. De Blasio, M. Hjorth-Jensen, Ø. Elgarøy, L. Engvik, G. Lazzari, M. Baldo and H.-J. Schulze, Phys. Rev. **C56**, 2332 (1997).
[14] D. Pines and M. A. Alpar, Nature **316**, 27 (1985).
[15] L. Kramer and W. Pesch, Z. Phys. **269**, 59 (1974.
[16] R. Epstein and G. Baym, Astrophys. J. **328**, 680 (1988).
[17] P. M. Pizzochero, L. Viverit and R. A. Broglia, Phys. Rev. Lett. **79**, 3347 (1997).
[18] F. V. De Blasio, Ø. Elgarøy, L. Engvik, and M. Hjorth-Jensen, in preparation.

k_F (fm^{-1})	Δ_∞	ξ_0 (fm)	ξ_2 (fm)
0.1	0.06	26.80	11.46
0.2	0.38	8.73	6.59
0.5	1.91	4.42	10.20

TABLE I. Gap at infinity and the length scales ξ_0 and ξ_2 as functions of density.

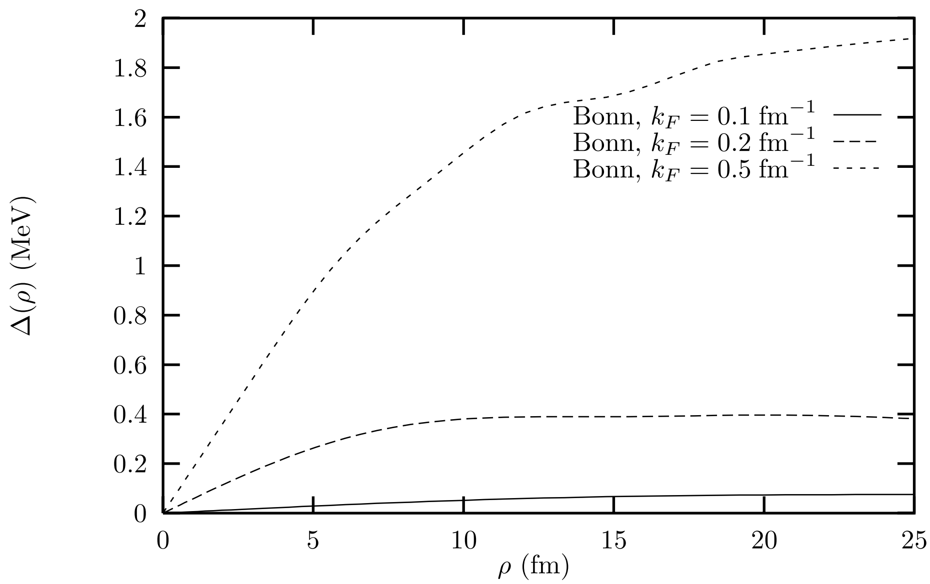

FIG. 1. Pairing potential $\Delta(\rho)$ as a function of ρ for an isolated vortex in neutron matter.

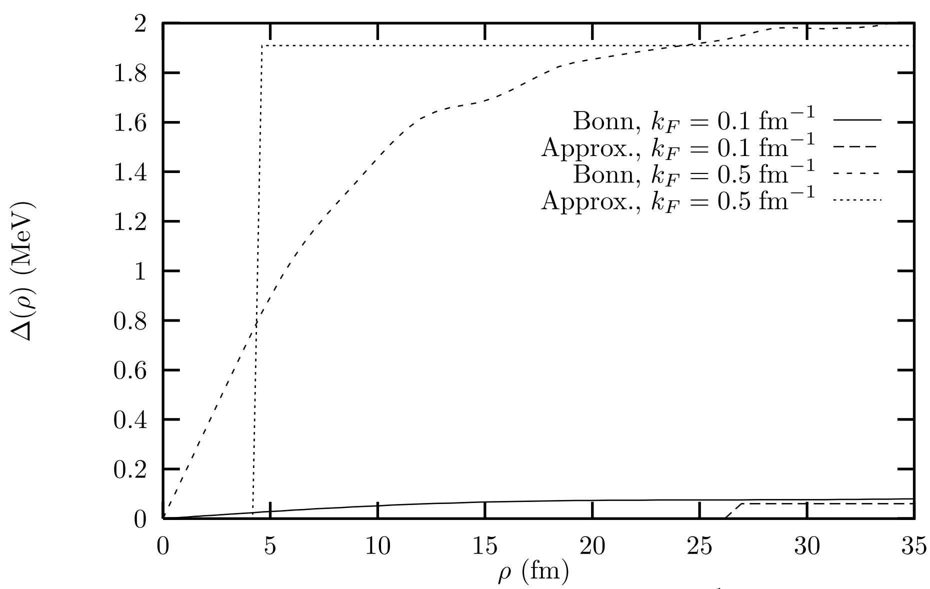

FIG. 2. The exact pairing potentials at $k_F = 0.1$ and 0.5 fm^{-1} compared with those obtained by using the ξ_0 cutoff prescription described in the text.